A theorem of Bernstein type for minimal surfaces in $\mathbb{R}^4$

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学位審査報告

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（学位論文題目）

A theorem of Bernstein type for minimal surfaces in R^4

（R^4内の極小曲面に対するベルンシュタイン型定理）

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（論文内容の要旨）

n次元ユークリッド空間R^n内の曲面は，その面積が曲面の微小な変形に関して極値をとるとき　極小曲面であるといわれる。

極小曲面に関する古典的な Bernstein の定理は，「可微分関数 f：R^2→RのグラフがR^3内の極小曲面であれば，fは1次関数である」と述べられる。

この定理の高次元化は，関数 f：R^n→Rのグラフの極小性として議論され各種の結果がえられているが，この場合グラフで与えられる極小部分多様体は余次元1のものである。

一方，複素数体CをR^2とみて，整関数：C→Cを可微分写像 f：R^2→R^2と考えると，そのグラフはR^4内の極小曲面である。この例は，余次元が1より大きい場合に Bernstein の定理を拡張するとき，fとして1次写像以外のものが現れ，余次元1の場合に比してその様相が複雑化することを示している。

申請者は，4次元ユークリッド空間R^4内の極小曲面の研究を続けてきたが，最近の do Carmo-Peng，Fisher-Colbrie による Bernstein の定理の一般化「R^3内の完備可符号安定極小曲面は平面である」において，極小曲面の安定性つまり第2変分が正または0となることが極小曲面の考察における重要な因子であることに着目し，次の定理を主論文において証明した。

定理　可微分写像 f：R^2→R^2 のグラフが R^4 内の安定極小曲面であればfは次のいずれかである。

(1) 1次写像　(2) 複素解析関数　(3) 反複素解析関数

申請者は以下の様な方法で定理を証明した。

極小曲面の法バンドルの2つの断面に関する第2変分の和を変形すると，平面上のコンパクトな台をもつ関数に作用する楕円型作用素がえられる。

定理の(1)，(2)，(3)のいずれでもない f に対して，この作用素が負の固有値をもつことを示す。その結果グラフの極小曲面の第２変分が負となって，この曲面は安定でないことがわかる。

申請者の方法は，余次元が２以上の極小曲面の不安定性を示す一般的方法としては，最初のものである。

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（論文審査の結果の要旨）

申請者は余次元が2以上の極小曲面の不安定性を示す一般的方法を研究し，すでに相当の成果を挙げている。

この場合の考察の難かしさは，曲面の法バンドルが一般には複雑で，法バンドルの断面に作用する楕円型作用素が取り扱い難いところにある。

現在までに知られている方法が適用できるのは，曲面を含む空間が特に単純で楕円型作用素を取り扱わずにすむ場合か，あるいは余次元が1で作用素が関数に作用する場合かのいずれかである。これに比して申請者の方法は，適当な断面をとることによって関数に作用する作用素の問題に帰着させるものであって，応用範囲の広いものである。

主論文における定理は，この方法の応用として，古典的な Bernstein の定理に対応する余次元2の場合の事実を示したものであり，この方面の研究において一つの画期的成果であるといえる。

なお，3篇の参考論文は，上に述べた方法をいくつかの重要な場合に応用したものであり，申請者のこの方面の研究発展に対する寄与と十分な学識を示しているものである。

以上によって，本論文は理学博士の学位を授与する価値あるものと認める。

なお，主論文及び参考論文に報告されている研究業績を中心とし，これに関連した研究分野について試問した結果，合格と認めた。

A theorem of Bernstein type for minimal surfaces in $\mathbb{R}^4$

SHIGEO KAWAI

1. Introduction. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a function the graph of which is a minimal surface in $\mathbb{R}^3$ Then u is a linear function by the classical theorem of Bernstein.

On the other hand, the same conclusion does not hold in the case of codimension greater than one. The graph of an entire holomorphic function on $\mathbb{C}$ is a minimal surface of $\mathbb{R}^4 = \mathbb{C}^2$ which is not necessarily a plane.

Recently a generalization of Bernstein's theorem was proved.

Theorem (do Carmo and Peng [1], Fischer-Colbrie and Schoen [3]) Complete orientable stable minimal surfaces in $\mathbb{R}^3$ are planes.

A minimal surface is called stable if the second variation is non-negative for every normal vector field on M with compact support. This theorem suggests that the essential property is the stability. In fact, the minimal graphs of codimension one are stable (Federer [2]), whereas those of codimension greater than one are not necessarily stable (Lawson and Osserman [5], Kawai [4])

Since the graph of a holomorphic function on $\mathbb{C}$ is stable, it is quite natural to ask whether or not the complete orientable stable minimal surfaces in $\mathbb{R}^4$ are congruent to the complex submanifolds of $\mathbb{C}^2 = \mathbb{R}^4$, i.e., transformed to the complex submanifolds of $\mathbb{C}^2 = \mathbb{R}^4$ by the isometries of $\mathbb{R}^4$

The purpose of this paper is to give a partial answer to this question, i.e., to prove the following theorem.

Theorem. Let M be a minimal surface in $\mathbb{R}^4$ which is a graph of a function defined on the whole plane $\mathbb{R}^2$ Suppose that M is stable. Then M is a plane or the graph of a holomorphic function or the graph of an antiholomorphic function with respect to a fixed identification $\mathbb{R}^2 = \mathbb{C}$. Hence M is congruent to a complex submanifold of $\mathbb{C}^2 = \mathbb{R}^4$

To prove this theorem, we shall show that the second variation is negative for some normal vector field on M with compact support if M is not of one of the three types. The description of minimal graphs in $\mathbb{R}^4$ by Osserman [6] will be used.

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2. The second variations. Let M be a surface minimally immersed in a flat Riemannian manifold N Suppose that M has global isothermal coordinates x, y, and the induced metric is $ds^2 = \lambda^2(dx^2 + dy^2)$ for a positive C^∞ function λ We make use of the same method as Kawai [4] But we

consider the complexified second variation instead of the sum of the second variations in the directions of a pair of normal vector-fields.

We write $X = \partial/\partial x$, $Y = \partial/\partial y$, $Z = (X - iY)/2$, $\overline{Z} = (X + iY)/2$ for simplicity, and denote various operators and their complexifications by the same letters. We denote by $\langle V,W \rangle$ the symmetric product of vectors V and W Hence $\|V\|^2 = \langle V,\overline{V} \rangle$ for a complex vector V

Proposition 1. Suppose a C^∞ cross-section ξ of the complexified normal bundle $\nu^{\mathbb{C}}$ of M satisfies the differential equation $\nabla_{\overline{Z}}\xi = 0$, where ∇ denotes the covariant differentiation in the normal bundle ν Then for every $\mathbb{R}$-valued C^∞ function φ on M with compact support, we have

$$I(\varphi\xi, \varphi\overline{\xi}) = 4\int_M (Z\varphi)(\overline{Z}\varphi)\|\xi\|^2\,dxdy - 4\int_M \|A^{\xi}(Z)\|^2\varphi^2\,dxdy,$$

where I is the index form, and A is the C^∞ cross-section of $\nu^*\otimes T^*M\otimes TM$ defined by the second fundamental form B of M in N

Proof. By the result of Simons [8], we have

$$I(\varphi\xi, \varphi\bar{\xi}) = \int_M [-\langle \Delta(\varphi\xi), \varphi\bar{\xi}\rangle - \langle A^{\varphi\xi}, A^{\varphi\bar{\xi}}\rangle] *1,$$

where $\Delta = \text{trace}\,\nabla\nabla$ is the Laplacian in the normal bundle ν of M in N, and $*1$ is the volume form of the induced metric of M

Since x, y are isothermal coordinates, we get

$$\Delta(\varphi\xi) = (1/\lambda^2)\,[\nabla_X\nabla_X(\varphi\xi) + \nabla_Y\nabla_Y(\varphi\xi)]$$

$$= (2/\lambda^2)\,[\nabla_Z\nabla_{\bar{Z}}(\varphi\xi) + \nabla_{\bar{Z}}\nabla_Z(\varphi\xi)]$$

$$= (4/\lambda^2)\,\{(Z\bar{Z}\varphi)\xi + (\bar{Z}\varphi)\nabla_Z\xi\,]$$

$$-(2/\lambda^2)\varphi R(Z, \bar{Z})\xi,$$

where R denotes the curvature of the normal bundle ν Hence we have

$$-\langle \Delta(\varphi\xi), \varphi\bar{\xi}\rangle = -(4/\lambda^2)\,[(Z\bar{Z}\varphi)\varphi\|\xi\|^2 + \varphi(\bar{Z}\varphi)(Z\|\xi\|^2)]$$

$$+ (2/\lambda^2)\varphi^2\langle R(Z,\bar{Z})\xi, \bar{\xi}\rangle$$

$$= -(1/\lambda^2)\,[(\Delta_0\varphi)\varphi\|\xi\|^2 + \varphi(X\varphi)(X\|\xi\|^2)$$

$$+\varphi(Y\varphi)(Y\|\xi\|^2) + i\left\{(Y\varphi)(X\|\xi\|^2) - (X\varphi)(Y\|\xi\|^2)\right\}]$$

$$+ (2/\lambda^2)\varphi^2\langle R(Z,\bar{Z})\xi, \bar{\xi}\rangle,$$

where $\Delta_0 = XX + YY = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

By the theorem of Stokes, we get

$$\int_M - (1/\lambda^2)[(\Delta_0\varphi)\varphi\|\xi\|^2 + \varphi(X\varphi)(X\|\xi\|^2) + \varphi(Y\varphi)(Y\|\xi\|^2)$$

$$+ i\left\{(Y\varphi)(X\|\xi\|^2) - (X\varphi)(Y\|\xi\|^2)\right\}] *1$$

$$= \int_M [(X\varphi)^2 + (Y\varphi)^2]\|\xi\|^2 \, dxdy$$

By the identity of Ricci, we have

$$(2/\lambda^2)\varphi^2\langle R(Z,\bar{Z})\xi, \bar{\xi}\rangle - \langle A^{\varphi\xi}, A^{\varphi\bar{\xi}}\rangle$$

$$= (2/\lambda^2)\varphi^2 [\langle A^{\bar{\xi}}(Z), A^{\xi}(\bar{Z})\rangle - \langle A^{\xi}(Z), A^{\bar{\xi}}(\bar{Z})\rangle$$

$$- \langle A^{\xi}(Z), A^{\bar{\xi}}(\bar{Z})\rangle - \langle A^{\xi}(\bar{Z}), A^{\bar{\xi}}(Z)\rangle]$$

$$= - (4/\lambda^2)\varphi^2\|A^{\xi}(Z)\|^2$$

Hence we obtain the desired result.

3. Minimal surfaces in $\mathbb{R}^4$ Consider a minimal surface M in $\mathbb{R}^n$ defined by

$$f(x,y) = (f_1(x,y), \ldots, f_n(x,y))$$

with respect to isothermal coordinates x , y Let us define functions ϕ_k of $z = x + iy$ by

$$\phi_k = \partial f_k / \partial x - i\, \partial f_k / \partial y \qquad (k = 1, 2, \ldots , n)$$

Since x , y are isothermal coordinates, ϕ_k are holomorphic in z and they satisfy the following identities.

(1) $$\langle \phi, \phi \rangle = \phi_1{}^2 + \phi_2{}^2 + \ldots + \phi_n{}^2 = 0 ,$$

(2) $$\| \partial f / \partial x \|^2 = \| \partial f / \partial y \|^2 = \| \phi \|^2 / 2$$

$$= (|\phi_1|^2 + |\phi_2|^2 + \ldots + |\phi_n|^2) / 2$$

Hence the induced metric of M is $ds^2 = \lambda^2 (dx^2 + dy^2)$ with $\lambda^2 = \|\phi\|^2/2$

We identify $X, Y, Z, \overline{Z}$ with their image by the differential of f

Definition. $\xi_0 = B(X,X) - i\, B(X,Y)$

By Ruh [7], the normal vector fields $B(X,X)$ and $B(X,Y)$ satisfy the following identities

$$\nabla_Y (B(X,X)) = \nabla_X (B(X,Y)) , \quad \nabla_X (B(X,X)) + \nabla_Y (B(X,Y)) = 0$$

Hence ξ_0 satisfies the equation $\nabla_{\overline{Z}}\xi = 0$.

We shall construct a cross-section ξ of $\nu^{\mathbb{C}}$ from ξ_0 which also satisfies the equation $\nabla_{\overline{Z}}\xi = 0$, and apply Proposition 1 to ξ For this purpose we study the properties of ξ_0

Lemma 1. $\xi_0 = 2B(Z,Z) = Z\phi - (Z\lambda^2)\phi/\lambda^2$

Proof. By the minimality of M , we have $B(Z,\overline{Z}) = 0$ Hence we get

$$\xi_0 = B(Z+\overline{Z},\ Z+\overline{Z}) - iB(Z+\overline{Z},\ i(Z-\overline{Z})) = 2B(Z,Z)$$

Since the real (resp. imaginary) part of $B(Z,Z)$ is the normal component of the real (resp. imaginary) part of $ZZf = Z\phi/2$, and $\|Zf\|^2 = \|\overline{Z}f\|^2 = \lambda^2/2$, we have

$$B(Z,Z) = Z\phi/2 - 2\langle Z\phi/2,\ \overline{Z}f\rangle Zf/\lambda^2 - 2\langle Z\phi/2,\ Zf\rangle \overline{Z}f/\lambda^2$$

Making use of the identities (1) and (2), we obtain

$$B(Z,Z) = Z\phi/2 - (Z\lambda^2)\phi/2\lambda^2$$

Thus the Lemma is proved.

Lemma 2. $\|A^{\xi_0}(Z)\|^2 = |\langle z\phi, z\phi\rangle|^2/2\lambda^2$,

$$\|\xi_0\|^2 = \|z\phi\|^2 - (2/\lambda^2)|z\lambda^2|^2$$

Proof. By the definitions of A and ξ_0 , we have

$$(3)\qquad \|A^{\xi_0}(Z)\|^2 = (2/\lambda^2)|\langle A^{\xi_0}(Z), \overline{Z}\rangle|^2 + (2/\lambda^2)|\langle A^{\xi_0}(Z), Z\rangle|^2$$

$$= (2/\lambda^2)|\langle B(Z,\overline{Z}), \xi_0\rangle|^2 + (2/\lambda^2)|\langle B(Z,Z), \xi_0\rangle|^2$$

$$= (1/2\lambda^2)|\langle \xi_0, \xi_0\rangle|^2$$

By Lemma 1 and the identity (1), we get the first equality. By Lemma 1, we have

$$\|\xi_0\|^2 = \langle z\phi - (1/\lambda^2)(z\lambda^2)\phi, \overline{z\phi} - (1/\lambda^2)(\overline{z}\lambda^2)\overline{\phi}\rangle.$$

Since $\|\phi\|^2 = 2\lambda^2$, we get the second equality.

Now we consider the minimal surfaces in $\mathbb{R}^4$ which are graphs of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ These objects are investigated by Osserman [6]

Proposition 2 (Osserman). Let $M : f = (f_1, f_2, f_3, f_4)$ be a minimal surface in $\mathbb{R}^4$ where $f_3(f_1, f_2)$ and $f_4(f_1, f_2)$ are functions of f_1 and f_2 defined on the whole plane $\mathbb{R}^2$ Then there exists a linear transformation of $\mathbb{R}^2$

$$f_1 = x \ , \ f_2 = ax + by \quad (a, b \in \mathbb{R}, \ b > 0) \ ,$$

such that x , y are global isothermal coordinates of M

With respect to these isothermal coordinates, we have

$$\phi_1 = \partial f_1 / \partial x - i \partial f_1 / \partial y = 1 \ ,$$

$$\phi_2 = \partial f_2 / \partial x - i \partial f_2 / \partial y = a - ib$$

Putting $d = 1 + (a - ib)^2$, we get $\phi_3{}^2 + \phi_4{}^2 = -d$, since $\phi_1{}^2 + \phi_2{}^2 + \phi_3{}^2 + \phi_4{}^2 = 0$

In the case $d = 0$, we show that M is either the graph of a holomorphic function or the graph of an antiholomorphic function with respect to a fixed complex structure of $\mathbb{R}^2$ Because the condition $d = 0$ implies $a = 0$ and $b = \pm 1$, the linear transformation in Proposition 2 is either $f_1 + if_2 = x + iy$ or $f_1 + if_2 = x - iy$ The identity $\phi_3{}^2 + \phi_4{}^2 = 0$ implies that either $\phi_4 = i\phi_3$ or $\phi_4 = -i\phi_3$ Hence $f_3 + if_4$ is either a holomorphic function or an antiholomorphic function of $f_1 + if_2$

Of course M is congruent to a complex submanifold of $\mathbb{C}^2 = \mathbb{R}^4$.

Now we consider the case $d \neq 0$ In this case the condition

$$(\phi_3 + i\phi_4)(\phi_3 - i\phi_4) = -d \neq 0$$

implies that the non-vanishing holomorphic function $\phi_3 - i\phi_4$ is of the form $\phi_3 - i\phi_4 = e^{H(z)}$ where $H(z)$ is an entire function. Hence we get the identities

$$(4) \qquad \phi_3 = (e^{H(z)} - de^{-H(z)})/2 \ , \quad \phi_4 = i(e^{H(z)} + de^{-H(z)})/2 \ ,$$

$$(5) \qquad z\phi = (0, 0, -H'(e^{H} + de^{-H})/2 \ , \ iH'(e^{H} - de^{-H})/2)$$

By Lemma 2 and the above identity (4), we have the following proposition.

Proposition 3, $\left\| A^{\xi_0}(z) \right\|^2 = |d|^2 \, |H'|^4/2 \, \lambda^2$

Here we give a property of a minimal surface in $\mathbb{R}^4$ which is congruent to a complex submanifold in $\mathbb{C}^2 = \mathbb{R}^4$

Lemma 3. M is congruent to a complex submanifold in $\mathbb{C}^2 = \mathbb{R}^4$ if and only if $H' \equiv 0$ or $d = 0$ holds.

Proof. We show that $\left\| A^{\xi_0}(Z) \right\|^2$ vanishes identically if M is congruent to a complex submanifold. We may suppose that M is a complex submanifold. Let us denote by J the complex structure of $\mathbb{C}^2 = \mathbb{R}^4$ Then J induces the complex structure J' of M and we have

$$B\,(\,J'u,\ v\,) = B\,(\,u,\ J'v\,) = JB(\,u,\ v\,)$$

for every tangent vector u and v of M at $p \in M$

Since J is Hermitian, we get by Lemma 1 the equality

$$\begin{aligned}
\langle \xi_0,\ \xi_0 \rangle &= 4 \langle JB(\,Z,\ Z\,)\ ,\ JB(\,Z,\ Z\,) \rangle \\
&= 4 \langle B(J'Z,\ Z)\ ,\ B(J'Z,\ Z) \rangle \\
&= 4 \langle B(\,iZ,\ Z)\ ,\ B(\,iZ\ ,\ Z\,) \rangle \\
&= - \langle \xi_0,\ \xi_0 \rangle
\end{aligned}$$

This shows $\langle \xi_0,\ \xi_0 \rangle = 0$ which implies the desired result by the identity (3) The converse is clear, because $H' \equiv 0$ implies that M is a plane.

Proposition 5. $\left\| \xi_0 \right\|^2 = [(\,1 + a^2 + b^2\,) - 2b^2/\lambda^2\,]\,\left| H' \right|^2$

Proof. Since

$$\lambda^2 = [\ 1 + a^2 + b^2 + (\ |e^H|^2 + |d|^2\ |e^{-H}|^2\ )/2\ ]/2\ ,$$

we have

$$|e^H|^2 + |d|^2\ |e^{-H}|^2 = 2\ [\ 2\lambda^2 - (\ 1 + a^2 + b^2\ )\ ]\ ,$$

$$(\ |e^H|^2 - |d|^2\ |e^{-H}|^2\ )^2 = 4\ [\ 2\lambda^2 - (\ 1 + a^2 + b^2\ )\ ]^2 - 4|d|^2$$

Hence we get

$$(6) \qquad z\lambda^2 = H'\ (\ |e^H|^2 - |d|^2\ |e^{-H}|^2\ )/4\ ,$$

$$2\ \left|z\lambda^2\right|^2/\lambda^2 = |H'|^2\ [\ \{2\lambda^2 - (\ 1 + a^2 + b^2\ )\}^2 - |d|^2\ ]/2\lambda^2$$

By Lemma 2, we obtain

$$\|\xi_0\|^2 = |H'|^2\ |\ e^H + de^{-H}|^2\ /4 + |H'|^2\ |\ e^H - de^{-H}|^2\ /\ 4$$

$$- |H'|^2\ [\ \{2\lambda^2 - (\ 1 + a^2 + b^2\ )\}^2 - |d|^2\ ]\ /2\lambda^2$$

$$= |H'|^2[(\ 1 + a^2 + b^2\ ) - 2\bar{b}/\lambda^2\ ]$$

4. The proof of Theorem. In this section, we show under the conditions $d \neq 0$ and $H' \not\equiv 0$ the existence of a normal vector field with compact support for which the second variation is negative. By Lemma 1 and the identities (5) and (6), we have

$$\xi_0 = H' \,[(\,0,\ 0,\ (\,e^H + de^{-H}\,)/2,\ i(\,e^H - de^{-H}\,)/2\,)$$

$$- (1/4\lambda^2)(\,|e^H|^2 - |d|^2|e^{-H}|^2\,)\ \phi\,]$$

Definition. We define a C^∞ cross section ξ of $\nu^{\mathbb{C}}$ to be the quantity inside the bracket [ ] in the above expression for ξ_0

Lemma 4. Suppose that $H' \not\equiv 0$ Then we have the following equalities.

$$\nabla_{\bar{Z}}\xi = 0, \quad \|A^{\xi}(Z)\|^2 = |d|^2\,|H'|^2\,/2\lambda^2\ ,$$

$$\|\xi\|^2 = (\,1 + a^2 + b^2\,) - 2b^2/\lambda^2 \leq 1 + a^2 + b^2$$

Proof. Since $\xi = \xi_0/H'$ at the points z with $H'(z) \neq 0$, we get the second and the third equalities. Since H' is holomorphic, we get the first equality except at the zero points of H' These equalities hold on the whole plane $\mathbb{R}^2$, because the zero points of H' are isolated.

By Proposition 1 and the above Lemma 4, we have

$$(7) \qquad I(\varphi\xi, \varphi\bar{\xi}) \leq (1 + a^2 + b^2)\int_{\mathbb{R}^2} [(X\varphi)^2 + (Y\varphi)^2]\, dxdy$$
$$- 2|d|^2 \int_{\mathbb{R}^2} (|H'|^2/\lambda^2)\, \varphi^2\, dxdy$$

for every $\mathbb{R}$-valued C^∞ function φ on $\mathbb{R}^2$ with compact support. Since

$$I(\varphi\xi, \varphi\bar{\xi}) = I(\varphi \operatorname{Re}\xi, \varphi \operatorname{Re}\xi) + I(\varphi \operatorname{Im}\xi, \varphi \operatorname{Im}\xi)$$

is the sum of the second variations in the directions of $\varphi\operatorname{Re}\xi$ and $\varphi \operatorname{Im}\xi$, we have only to show the existence of an $\mathbb{R}$-valued C^∞ function φ with compact support for which the right hand side of the inequality (7) is negative.

The quantity $|H'|^2$ is positive on an open set of $\mathbb{R}^2$ Hence it suffices to prove the following lemma.

Lemma 5. Let c be a positive constant. Let F be a non-negative function on $\mathbb{R}^2$ which is positive on an open neighborhood of the origin. Then there exists an $\mathbb{R}$-valued C^∞ function φ on $\mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} [(X\varphi)^2 + (Y\varphi)^2]\, dxdy < c\int_{\mathbb{R}^2} F\varphi^2\, dxdy$$

Proof. Let us define a sequence φ_m ($m = 1, 2, \ldots$) of functions on $\mathbb{R}^2$ as follows:

$$\varphi_m(r,\theta) = 1/3 + 1/5 + \dots + 1/(2m-1) \qquad (0 \lneqq r \leqq 1)$$

$$\varphi_m(r,\theta) = 1/(2j+1) + 1/(2j+3) + \dots + 1/(2m-1) - (r-j)/(2j+1)$$

$$(j \leqq r \leqq j+1, \quad j = 1,2,\dots,m-1) ,$$

$$\varphi_m(r,\theta) = 0 \qquad (m \leqq r) ,$$

where r, θ are the polar coordinates of $\mathbb{R}^2$ Then we have

$$\int_{\mathbb{R}^2} [(X\varphi_m)^2 + (Y\varphi_m)^2] \, dxdy = \pi [1/3 + 1/5 + \dots + 1/(2m-1)] ,$$

$$c \int_{\mathbb{R}^2} (F\varphi_m{}^2) \, dxdy \geq c \int_{D(1)} (F\varphi_m{}^2) \, dxdy$$

$$= c' [1/3 + 1/5 + \dots + 1/(2m-1)]^2$$

where $D(1) = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \leqq 1 \right\}$ and $c' = c \int_{D(1)} F \, dxdy > 0$

Since $1/3 + 1/5 + \dots 1/(2m-1) \longrightarrow \infty$ ($m \longrightarrow \infty$) , we have

$$\int_{\mathbb{R}^2} [(X\varphi_m{}^2) + (Y\varphi_m{}^2)] \, dxdy < c \int_{\mathbb{R}^2} (F\varphi_m{}^2) \, dxdy$$

for sufficiently large m Approximating φ_m by a C^∞ function, we obtain the desired result.

Remark. The first proof of Lemma 5 used the following fact due to Fischer-Colbrie and Schoen [3] For any non-negative function q, the existence of a positive function g on $\mathbb{R}^2$ satisfying $-\Delta_0 g - q\cdot g = 0$ is equivalent to the condition that the first eigenvalue of $-\Delta_0 - q$ be non-negative on each bounded domain in $\mathbb{R}^2$ The above elementary proof was kindly informed to the author by Professor K. Sugahara.

Remark. For a orientable parametric minimal surface M in $\mathbb{R}^4$, the following can be proved: M is congruent to a complex submanifold in $\mathbb{C}^2 = \mathbb{R}^4$ if and only if $\|A^{\xi_0}(Z)\|$ vanishes identically, i.e., $\langle \xi_0, \xi_0 \rangle$ vanishes identically on the domain of an isothermal coordinate. This fact may be useful to generalize Theorem for parametric minimal surfaces in $\mathbb{R}^4$

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